

The Candy-Passing Game for $c \geq 3n - 2$

Paul M. Kominers*

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Abstract

We determine the behavior of Tanton's candy-passing game for all distributions of at least $3n - 2$ candies, where n is the number of students. Specifically, we show that the configuration of candy in such a game eventually becomes fixed.

The candy-passing game, as introduced by Tanton [1], is played according to the following rules:

- At the beginning of the game, $c > 0$ candies are distributed arbitrarily among $n > 2$ students, who are sitting in a circle.
- A whistle is sounded at a regular interval.
- Each time the whistle is sounded, each student with two or more candies passes one candy to his left-hand neighbor and one candy to his right-hand neighbor.
- If a student has fewer than two candies when the whistle sounds, he does nothing.

When $c < n$, the game eventually terminates, with no students having sufficient candy to pass candy to their neighbors (see [1]). In this paper, we study the behavior of the candy-passing game for $c \geq 3n - 2$, showing that the configuration of candy in such a game eventually becomes fixed.

*Walt Whitman High School, Bethesda, MD

We call the interval between blows of the whistle a *round* of candy-passing. The students are consecutively numbered $1, 2, \dots, k, \dots, n$, starting from an arbitrary student. The candy pile of a student having four or more pieces of candy is said to be *abundant*, and we denote the number of students with abundant candy piles by m . If, after some round, the amount of candy a student has will not change over the remainder of the given candy-passing game, that student's candy pile is said to have *stabilized*.

Clearly, if a student has two or more candies at the beginning of a round, that student cannot end the round with more candy than he began with. Indeed, in any round, a given student with two or more candies can, at most, pass two pieces of candy to his neighbors and receive two pieces of candy from his neighbors, resulting in no net increase in the size of his candy pile.

Lemma 1. *After a finite number of rounds, the set of students with abundant candy piles in any candy-passing game is fixed and the candy piles of all such students have stabilized.*

Proof. If $m = 0$ at the beginning of the game, there are no students with abundant candy piles to lose candy. We now assume that $m > 0$ at the beginning of the game. As we observed, the total amount of candy possessed by students with abundant candy piles is nonincreasing. Further, if a student with an abundant candy pile loses candy, that sum decreases. Since the total amount of candy possessed by students with abundant candy piles cannot fall below zero, the amount of candy that can be lost by students with abundant candy piles must be finite. \square

We are now ready to prove our main result.

Theorem 1. *In a candy-passing game with $c \geq 3n - 2$, then all students' candy piles eventually stabilize.*

Proof. As a consequence of Lemma 1, we may assume that all candy that may be lost by students with abundant candy piles has been lost, as this must happen within finitely many rounds. If $m = 0$ at this point, then the condition $c \geq 3n - 2$ implies $c = 3n$, $c = 3n - 1$, or $c = 3n - 2$.

If $m = 0$ and $c = 3n$, each student has three candies. If $m = 0$ and $c = 3n - 1$, each student has three candies except for one, who has two. In both of these cases, all of the students' candy piles have stabilized.

If $m = 0$ and $c = 3n - 2$, either each student has three candies except for two students who have two candies each, or each student has three candies

except for one student who has only one candy. In the first of these cases, all of the students' candy piles have stabilized. In the second, the neighbors of the student having only one candy each pass him one candy and receive one, reducing this situation to the first case.

We now assume $m > 0$. Since a student with an abundant candy pile passes candy each round, in order for his candy pile to have stabilized, he must be receiving candy from both of his neighbors every round. Select one student in the game with an abundant candy pile. Since he must pass candy every round, he must receive candy from both of his neighbors every round, who must therefore themselves have at least two pieces of candy every round. The neighbors must therefore eventually stabilize; there is a minimum amount of candy (two pieces) such that they can pass candy every round. They cannot drop below this number, or the abundant candy piles would not have stabilized. They cannot oscillate between various amounts of candy greater than two (say, between two and three), as any student with two or more pieces of candy cannot end the round with more candy than he began with. For them to have stabilized while passing candy every round, their neighbors must be passing candy every round, which means that they, too, must eventually stabilize. We see by this argument that for an abundant candy pile to have stabilized, all other candy piles must eventually stabilize. \square

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References

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